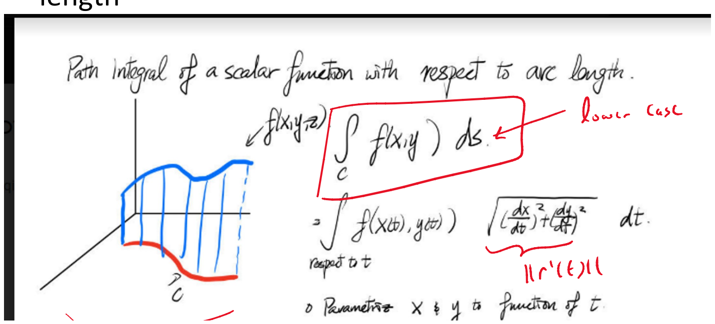
Math 53, Discussion 107

Wednesday, April 29, 2020 9:19 AM

Review of chapter 16 integrals

For each of the following, write a definition, formula, interpretation, and example

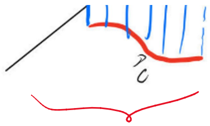
1) Path integral of a scalar function with respect to arc length



ds arc length

dS surface

$d\vec{S}$ surface w/ normal



$= \int f(x(t), y(t)) \underbrace{\sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2}}_{\|r'(t)\|}$ ...

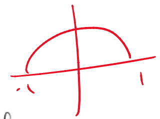
respect to t

∘ Parametrize x & y to function of t.

⇓

one variable calc.

Example: $f(x,y) = x^2 y$.



C is a semicircle has bounds from −1 to 1.

$$r(t) = \begin{cases} x = \cos t \\ y = \sin t \end{cases} \quad 0 \le t \le \pi$$

$$\int_C f(x,y)\, ds = \int_0^{\pi} \underbrace{\cos^2 t \sin t}_{f(r(t))} \sqrt{(-\sin t)^2 + (\cos t)^2}\, dt$$

1) Path integral of a scalar function with respect to x or y

$$\int_C P\,dx + Q\,dy$$

<u>Def</u> $$\int_C f\, dx \equiv \int_a^b f(r(t))\, x'(t)\, dt$$

with $r = (x(t), y(t))$ param of x

1) Path integral of a vector function

$\vec{F} = \langle P, Q, R \rangle$

<u>Def</u> $$\int_C \vec{F} \cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt = \int_a^b P(r(t))\, x'(t)\,dt + Q(r(t))\, y'(t)\,dt + R(r(t))\, z'(t)\,dt$$

$+ R(r(t))z'(t)dt$

r param C

Interpretation: adding up vector field F along a curve C, where at each point, we consider how much our vector field agrees with C

1) Fundamental theorem of line integrals

<u>Def</u> $\int_C \nabla f \cdot d\vec{r} = f(r(b)) - f(r(a))$

Tool to compute path integrals if our vector field is conservative.

F is conservative $\iff \frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$ for a simply connected domain

$\Rightarrow$ Easy

$\Leftarrow$ Hard

1) Greens theorem

$$\int_C \vec{F} \cdot dr = \int_C P\,dx + Q\,dy = \iint_A \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

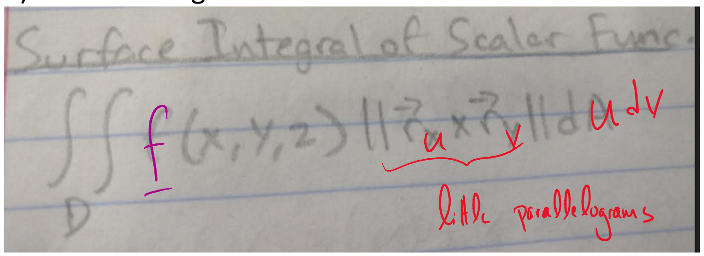
Closed, positively oriented

interior of C

This is a tool to compute 2d path integrals of vectors fields when they are not conservative

This is a lower dimensional version of Stokes' theorem

1) Curl
2) Divergence
3) Surface integral of scalar function

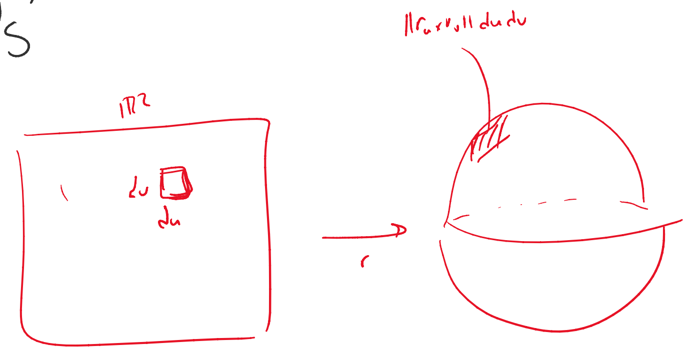


Surface Integral of Scalar Func.

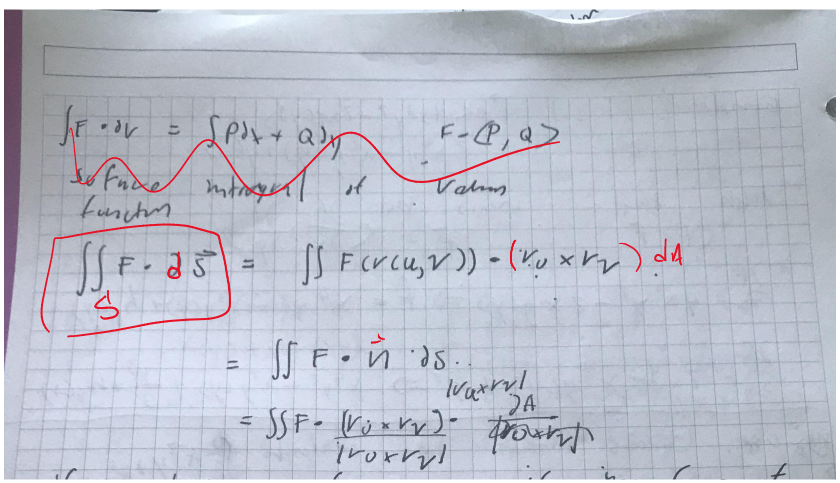
$$\iint_D \underline{f}(x,y,z)\,\underbrace{\|\vec{r}_u \times \vec{r}_v\|\,du\,dv}_{\text{little parallelograms}}$$

$$= \iint_S f\,dS$$

$r =$ param of S



1) Surface integral of vector function



$\cancel{\int F \cdot dr = \int P\,dx + Q\,dy \qquad F = \langle P, Q\rangle}$

~~surface function integral of values~~

$$\boxed{\iint_S F \cdot d\vec{S}} = \iint F(r(u,v)) \cdot (r_u \times r_v)\,dA$$

$$= \iint F \cdot \vec{n}\,dS$$

$$= \iint F \cdot \frac{(r_u \times r_v)}{|r_u \times r_v|} \cdot \frac{|r_u \times r_v|\,dA}{\cancel{|r_u \times r_v|}}$$

Interp: flux of F through surface S

r param of S

$\vec{n}$ = unit normal

$$= \iint F \cdot \frac{(r_u \times r_v)}{|r_u \times r_v|} \cdot |r_u \times r_v| \, dA$$

If the surface is of the form of $z = g(x,y)$

$r(x,y) = (x, y, g(x,y))$

$r_x = (1, 0, \frac{\partial g}{\partial x})$

$r_y = (0, 1, \frac{\partial g}{\partial y})$

~~$\iint \vec{F} \cdot d\vec{S}$~~

~~$f''(x)$~~

$$\begin{tikzcd} \int & \iint \arrow[l, bend left, "\text{Stokes}"] & \iiint \arrow[l, bend left, "\text{divergence}"] \end{tikzcd}$$

1) Stokes' theorem

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_S (\nabla \times \vec{F}) \cdot d\vec{S}$$

C a closed path and S <u>any</u> surface w/ boundary C

1) Divergence theorem

$$\iint_S \vec{F} \cdot d\vec{S} = \iiint_E \nabla \cdot \vec{F} \, dV$$

S = closed surface with interior E